# Scattering Losses in a Large Luneberg Lens Due to Random Dielectric Inhomogeneities

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*An approximate theoretical calculation is made of the scattering losses due to random inhomogeneities in a large Luneberg lens made of dielectric blocks. A simple model is used in which the average index of refraction of each block may differ slightly from the value called for by the lens design, and the index may vary linearly with position inside a given block. The losses are described by attributing an effective scattering loss tangent to the lens medium. Tables and curves are given to facilitate the computation of total scattering loss as a function of block size, mean-square deviation of average index, and mean-square index gradient within the blocks.*

*Manufacturing processes for foam dielectric blocks are monitored by testing the blocks in various orientations in an oversize resonant cavity. An approximate relationship is derived between the results of cavity resonator measurements and the parameters of the theoretical model; but it is pointed out that the assumption of linear index variation across a single block can be quite unrealistic in practice. Numerical results derived from the present theory are most likely to be meaningful if the dimensions of the individual blocks are less than one wavelength.*

## I. INTRODUCTION

In recent years, large Luneberg lenses have been used as antennas for long-range radars.[1] Such lenses have been built of cubical blocks of very low density polystyrene foam, loaded with varying amounts of aluminum slivers[2] in order to approximate the desired variation of refractive index between the surface of the lens and the center. In theory each block is perfectly homogeneous and isotropic and has a specified permittivity; in practice, however, the blocks are not perfectly homogeneous or isotropic, and the average permittivity of a block generally differs more or less from the value called for by the designer. If the tolerances on the

blocks are too loose, excessive power will be lost from the lens by scattering from the "misfits"; in extreme cases, scattering may also lead to unacceptable antenna patterns. On the other hand, if the tolerances are too tight the yield of acceptable blocks by any reasonable manufacturing process will be reduced, and the cost of the lens correspondingly increased. Considerable importance attaches, therefore, to setting the proper tolerances.

This paper treats the problem of scattering losses in a large Luneberg lens, due to random dielectric inhomogeneities, on the basis of a simple mathematical model which is described in Section II. According to the present analysis, the effect of dielectric scattering can be described by attributing to the material of the lens an effective loss tangent given by (1) of Section II. The expression for the effective loss tangent is derived in Section III. Section IV discusses the relationships between the dielectric deviations whose mean-square values appear in (1) and the results of resonant cavity measurements on individual blocks, and stresses that real blocks may not be very well represented by the idealized model. Finally, a few illustrative numerical examples are worked out in Section V.

## II. DESCRIPTION OF THE MODEL

We consider a "block" lens, that is, a structure built of cubical dielectric blocks which are intended to approximate an ideal, spherically symmetric Luneberg lens with a continuously varying index of refraction. The analysis is based on the following assumptions.

(*i*) We suppose that a nominal design for a block lens is given. We do not attempt to decide how many different index values are necessary, or how large the individual blocks can be. We merely assume that the electrical performance of the lens would be satisfactory if all blocks were perfectly uniform and homogeneous and had exactly the specified refractive indices; and we investigate how much the performance of the lens would be degraded by deviations in the dielectric properties of the blocks.

(*ii*) We assume that the permittivity of each block is in fact a linear function of rectangular coordinates in the block, and that the average permittivity (which in the linear model occurs at the center of the block) may differ slightly from the value assigned to the block in the nominal design.

(*iii*) Since the lens is many wavelengths in diameter, we assume that the power scattered out of the lens by a particular block is equal to the power which would be scattered from a plane wave by a similar block in

an infinite, uniform medium having the dielectric properties of the nominal block at that point. We further assume that the scattering pattern of a single block is so much broader than the beamwidth of the lens that all the scattered power may be considered lost.

(*iv*) We assume that the deviations in permittivity are completely uncorrelated from one block to the next, and that the direction of dielectric gradient is uniformly distributed over all angles. The assumption of statistical independence permits us to add the scattered power from each block directly, instead of adding complex amplitudes and then squaring, as we would have to do if the properties of a given block were correlated with those of its neighbors.

(*v*) We approximate the scattering from a cubical block by the scattering from a spherical "blob" of equal volume and similar dielectric properties. This is certainly as accurate as the other approximations involved in the model, and it reduces the mathematical problem essentially to one which has already been solved in the theory of tropospheric scattering.[3,4]

Under the foregoing assumptions and approximations, it turns out that the power lost by scattering from random dielectric inhomogeneities in a medium built of cubical blocks may be represented by an effective loss tangent:

$$\tan \delta = 3 \frac{\langle (\Delta n_0)^2 \rangle}{n_0^2} \varphi_0(x) + \frac{16}{9} \frac{\langle (\Delta n_1)^2 \rangle}{n_0^2} \varphi_1(x). \tag{1}$$

The symbols on the right side of (1) are defined as follows:

$n_0$ : nominal refractive index of a given block;

$\Delta n_0$ : difference between average index of a given block and nominal index;

$\Delta n_1$ : difference between average indices of the "heaviest" and "lightest" halves of a linearly-varying block (the direction of index gradient need not, of course, be parallel to any edge of the block);

$x = 2\pi a n_0/\lambda_v$ , where $\lambda_v$ is the vacuum wavelength;

$a = 0.620l$: radius of a sphere whose volume is equal to that of a cube of edge $l$;

$\varphi_0(x), \varphi_1(x)$: functions defined by (30) and (31), tabulated in Table I, and plotted in Figs. 1 and 2;

$\langle\ \rangle$: average over the neighborhood of a given block.

It must be emphasized that (1) is *not* applicable to the effects of *systematic* deviations from the ideal Luneberg lens structure. For example, one might be tempted to apply it to calculate how large the individual blocks could be, if each block were perfectly uniform and

had the index of refraction called for by the ideal Luneberg law at its center. In that case the index of the part of each block nearest the center of the lens would be systematically too low, while the index of the farthest part would be systematically too high. This would violate the assumption of block-to-block randomness, on the basis of which we added the scattered power to obtain (1). The equation therefore cannot be used to deduce the allowable coarseness of the nominal design. It can only tell us the effects of *random* variations in a nominal design which is believed, on other grounds, to be satisfactory.

## III. SCATTERING BY A "SOFT" SPHERICAL BLOB

In this section we shall compute approximately the power scattered out of an incident plane wave by a "soft" spherical blob, that is, a spherical region whose permittivity differs but little from the permittivity of the uniform surrounding medium. This kind of scattering is often called Rayleigh-Gans scattering and has an extensive literature.[5] We shall briefly derive the specific results that we need.

Consider a blob of radius $a$, centered at the origin of the spherical coordinate system $(r,\theta,\varphi)$. The permittivity of the blob is taken to be

$$\epsilon(\mathbf{r}) = \epsilon_0 + \epsilon_1(\mathbf{r}), \tag{2}$$

where $\epsilon_0$ is the permittivity of the surrounding medium, and

$$| \epsilon_1(\mathbf{r})/\epsilon_0 | \ll 1. \tag{3}$$

For an anisotropic blob $\epsilon_1(\mathbf{r})$ would be a tensor function, but we shall not consider this additional complication. A linearly polarized plane wave, whose electric field is given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \exp(-i\beta\mathbf{k}'\cdot\mathbf{r}), \tag{4}$$

is incident upon the blob. Here $\mathbf{E}_0$ is a constant vector, $\mathbf{k}'$ is a unit vector in the direction of propagation, and $\beta = 2\pi/\lambda_0$, where $\lambda_0$ is the wavelength of a free wave in the surrounding medium. The time dependence $e^{i\omega t}$ is understood throughout.

According to the basic Rayleigh-Gans approximation, a typical differential volume element $d\mathbf{r}'$ at $\mathbf{r}'$ scatters as if it were immersed in a uniform medium of permittivity $\epsilon_0$ and had an electric dipole moment

$$\mathbf{dp} = i\omega\epsilon_1(\mathbf{r}')\mathbf{E}(\mathbf{r}')d\mathbf{r}'. \tag{5}$$

We wish to describe the scattered field at the point $\mathbf{r} = \mathbf{k}r$ in the far zone, where $\mathbf{k}$ is a unit vector in an arbitrary direction. For this purpose

we define[6] the magnetic radiation vector $\mathbf{N}$. The contribution to $\mathbf{N}$ from a typical scattering element is

$$d\mathbf{N} = \exp(i\beta\mathbf{k}\cdot\mathbf{r}')\, d\mathbf{p} = i\omega\epsilon_1(\mathbf{r}')\mathbf{E}_0 \exp[i\beta(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}']\, d\mathbf{r}', \quad (6)$$

so the total magnetic radiation vector in a given scattering direction is

$$\mathbf{N}(\mathbf{k} - \mathbf{k}') = i\omega\mathbf{E}_0 \int_V \epsilon_1(\mathbf{r}') \exp[i\beta(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}']\, d\mathbf{r}', \quad (7)$$

where the integration is taken over the whole spherical scattering volume $V$.

So far the dependence of the "dielectric deviation" $\epsilon_1(\mathbf{r}')$ on position has not been specified. We now introduce the assumption that $\epsilon_1(\mathbf{r}')$ is a linear function of position with a mean value which possibly differs from zero. In symbols,

$$\epsilon_1(\mathbf{r}') = \epsilon_0\left[a_0 + \frac{a_1}{a}\,\mathbf{n}\cdot\mathbf{r}'\right], \quad (8)$$

where $a_0$ and $a_1$ are real numbers whose magnitudes are small compared to unity, $a$ is the radius of the sphere, and $\mathbf{n}$ is a unit vector in an arbitrary direction. The expression (7) then becomes

$$\mathbf{N}(\mathbf{k} - \mathbf{k}') = i\omega\epsilon_0\mathbf{E}_0 V[a_0 I_0 + a_1 I_1], \quad (9)$$

where

$$I_0 = \frac{1}{V}\int_V \exp[i\beta(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}']\, d\mathbf{r}', \quad (10)$$

$$I_1 = \frac{1}{aV}\int_V \mathbf{n}\cdot\mathbf{r}' \exp[i\beta(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}']\, d\mathbf{r}', \quad (11)$$

and

$$V = 4\pi a^3/3. \quad (12)$$

We are at liberty to choose any convenient coordinate system to describe the scattering problem. We take the $z$-axis in the direction of propagation $\mathbf{k}'$ of the incident wave, and the $x$-axis parallel to $\mathbf{E}_0$. The angular coordinates of the scattering direction $\mathbf{k}$ are denoted by $(\theta,\varphi)$.

In order to evaluate the integrals $I_0$ and $I_1$, we take advantage of the spherical symmetry of the scattering volume and introduce a new set of angular coordinates $(\chi,\alpha)$, with the new polar axis $\chi = 0$ along the vector $\mathbf{k} - \mathbf{k}'$. The plane $\alpha = 0$ is defined by the two vectors $\mathbf{k} - \mathbf{k}'$ and $\mathbf{n}$, so that the angular coordinates of $\mathbf{n}$ in this system are $(\chi_0, 0)$.

Noting that $|\mathbf{k} - \mathbf{k}'| = 2 \sin \frac{1}{2}\theta$, we have by a straightforward integration,

$$\begin{aligned} I_0 &= \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \exp\,(2i\beta r' \sin \tfrac{1}{2}\theta \cos \chi) r'^2 \sin \chi \, dr' \, d\chi \, d\alpha \\ &= G_0(2\beta a \sin \tfrac{1}{2}\theta), \end{aligned} \tag{13}$$

where

$$G_0(u) = (3/u^3)[\sin u - u \cos u]. \tag{14}$$

Similarly,

$$\begin{aligned} I_1 &= \frac{1}{aV} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} [\sin \chi_0 \sin \chi \cos \alpha + \cos \chi_0 \cos \chi] \\ &\qquad \times \exp\,(2i\beta r' \sin \tfrac{1}{2}\theta \cos \chi) r'^3 \sin \chi \, dr' \, d\chi \, d\alpha \\ &= G_1(2\beta a \sin \tfrac{1}{2}\theta) \cos \chi_0 , \end{aligned} \tag{15}$$

where

$$G_1(u) = (3/u^4)[(3 - u^2) \sin u - 3u \cos u], \tag{16}$$

and $\chi_0$ is the angle between $\mathbf{n}$ and $\mathbf{k} - \mathbf{k}'$.

From (9), (13), and (15), the only component of $\mathbf{N}$ is the one parallel to the incident electric field, and it is given by

$$N_x = i\omega\epsilon_0 E_0 V[a_0 G_0(2\beta a \sin \tfrac{1}{2}\theta) + a_1 G_1(2\beta a \sin \tfrac{1}{2}\theta) \cos \chi_0]. \tag{17}$$

The scattered power per unit solid angle is[6]

$$\Phi = \frac{\eta_0}{8\lambda_c^2} |N_x|^2 [\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi], \tag{18}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the characteristic impedance of the medium. The total scattered energy is obtained by integrating $\Phi$ over all directions in space.

Ultimately, we are going to assume that all directions $\mathbf{n}$ of dielectric gradient are equally probable, and it will simplify the subsequent calculations to carry out the averaging over $\mathbf{n}$ first. We obtain

$$\begin{aligned} &\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} |N_x|^2 \sin \chi_0 \, d\chi_0 \, d\alpha \\ &\qquad = \omega^2 \epsilon_0^2 E_0^2 V^2 [a_0^2 G_0^2(2\beta a \sin \tfrac{1}{2}\theta) + \tfrac{1}{3} a_1^2 G_1^2(2\beta a \sin \tfrac{1}{2}\theta)]. \end{aligned} \tag{19}$$

If $\bar{\Phi}(\theta,\varphi)$ is the result of the preliminary averaging over $\mathbf{n}$, the average power scattered per unit volume of the spherical blob can be written

in the form

$$\frac{1}{V}\int_0^{2\pi}\int_0^{\pi} \bar{\Phi} \sin\theta \, d\theta \, d\varphi = \frac{\beta E_0^2}{8\eta_0}\left[3a_0^2\varphi_0(x) + a_1^2\varphi_1(x)\right], \tag{20}$$

where*

$$x = \beta a = 2\pi a/\lambda_0, \tag{21}$$

and

$$\varphi_0(x) = \frac{x^3}{9}\int_0^{\pi} G_0^2(2x \sin \tfrac{1}{2}\theta)(\cos^2\theta + 1) \sin\theta \, d\theta, \tag{22}$$

$$\varphi_1(x) = \frac{x^2}{9}\int_0^{\pi} G_1^2(2x \sin \tfrac{1}{2}\theta)(\cos^2\theta + 1) \sin\theta \, d\theta. \tag{23}$$

On the other hand, the average scattered power per unit volume may be written in terms of the incident power density $E_0^2/2\eta_0$ and an effective attenuation constant $\alpha$, or an effective loss angle $\delta$, as

$$\frac{2\alpha E_0^2}{2\eta_0} = \frac{2\pi \tan\delta}{\lambda_0}\frac{E_0^2}{2\eta_0}. \tag{24}$$

Comparing (20) and (24), we obtain for the effective loss tangent,

$$\tan\delta = \tfrac{1}{4}\left[3a_0^2\varphi_0(x) + a_1^2\varphi_1(x)\right]. \tag{25}$$

Since the dielectric deviations described by the constants $a_0$ and $a_1$ are assumed small, and since the index of refraction of a dielectric medium is just the square root of the relative permittivity, the refractive index of the sphere described by (2) and (8) is approximately

$$n = n_0\left[1 + \frac{a_0}{2} + \frac{a_1 \mathbf{n}\cdot\mathbf{r}}{2a}\right]. \tag{26}$$

The relative deviation of the average index of the sphere from the surrounding value $n_0$ is

$$\frac{\Delta n_0}{n_0} = \frac{a_0}{2}, \tag{27}$$

and the relative difference between the average indices of the "heaviest" and "lightest" halves of the sphere is

$$\frac{\Delta n_1}{n_0} = \frac{2}{2\pi a^3/3}\int_0^{2\pi}\int_0^{\frac{1}{2}\pi}\int_0^{a} \frac{a_1 r \cos\theta}{2a} r^2 \sin\theta \, dr \, d\theta \, d\varphi = \frac{3a_1}{8}. \tag{28}$$

* The parameter $x$ defined by (21) has nothing to do with the coordinate $x$, which will never appear in the same context.

Substituting (27) and (28) into (25) and averaging over a random collection of blobs gives

$$\tan \delta = 3 \frac{\langle(\Delta n_0)^2\rangle}{n_0^2} \varphi_0(x) + \frac{16}{9} \frac{\langle(\Delta n_1)^2\rangle}{n_0^2} \varphi_1(x), \tag{29}$$

which is just (1) of Section II.

The functions $\varphi_0(x)$ and $\varphi_1(x)$, which were defined by (22) and (23), may be expressed either as infinite series or in closed form; thus,

$$\begin{aligned} \varphi_0(x) &= \sum_{n=2}^{\infty} \frac{(-)^n (4x)^{2n-1}(n^2 - n + 2)}{2n(n+1)^2(2n)!} \\ &= \frac{5}{8x} + \frac{x}{2} - \frac{\sin 4x}{16x^2} - \frac{7}{64x^3}(1 - \cos 4x) \\ &\quad + \left(\frac{1}{8x^3} - \frac{1}{2x}\right) \operatorname{Cin} 4x, \end{aligned} \tag{30}$$

$$\begin{aligned} \varphi_1(x) &= \sum_{n=3}^{\infty} \frac{(-)^{n-1}(4x)^{2n-1}(n^2 - n + 2)}{2n(n+1)^2(2n)!} \left(\frac{n-2}{n+2}\right) \\ &= -\frac{3}{32x^5} - \frac{37}{64x^3} + \frac{7}{8x} + \frac{x}{6} + \left(\frac{3}{8x^3} - \frac{1}{2x}\right) \operatorname{Cin} 4x \\ &\quad + \left(\frac{3}{32x^5} - \frac{11}{64x^3}\right) \cos 4x + \left(\frac{3}{8x^4} - \frac{3}{16x^2}\right) \sin 4x, \end{aligned} \tag{31}$$

where

$$\operatorname{Cin} x = \int_0^x \frac{1 - \cos t}{t} \, dt. \tag{32}$$

The functions $\varphi_0(x)$ and $\varphi_1(x)$ are tabulated in Table I and plotted in Figs. 1 and 2 (note the difference in scale of the two figures).

## IV. RESONANT CAVITY TECHNIQUES FOR TESTING DIELECTRIC BLOCKS

In practice, the dielectric blocks are tested in an oversize resonant cavity[7] before being assembled into the lens. A typical cavity is shown in Fig. 3; the cavity dimensions are $l \times 2l \times 3l$, and a cubical block of edge $l$ is placed with one face against the center of one of the long sides of the cavity. If the block were perfectly uniform and isotropic, a single measurement of resonant frequency would suffice to determine its permittivity, with the aid of an experimental or theoretical calibration curve which could be derived once for all. The dissipation, which we are neglecting in the present paper, could be deduced from the change in $Q$

of the loaded cavity. Actually the frequency shift when a cubical block is placed in the cavity depends on the orientation of the block. Twelve orientations altogether are possible for a block in the location shown in Fig. 3, and from the measured frequency shifts one attempts to deduce something about the nonuniformity and/or anisotropy of the given block.

Ideally one would like to have a simple correlation between the results of cavity resonator measurements and the total power scattered by a given block in the lens. In a real block, however, the permittivity is a (possibly tensorial) function of position, having in principle an infinite number of degrees of freedom. A single measurement of frequency shift yields a weighted average of the dielectric deviations, according to (33) below. On the other hand, the total power scattered by the block in the lens is a different and nonlinear functional of the dielectric deviations, given by (7) and (18) of Section III. In the general case it is obviously impossible to write the scattered power as a function of the results of any finite number of resonant cavity measurements.

In order to make any sort of connection between cavity resonator experiments and the performance of blocks in the lens, one has to adopt some model of the dielectric deviations in the blocks, describe the model in terms of a small number of parameters, express the parameters in terms of the results of cavity measurements, and finally calculate the scattered power as a function of the parameters. The parameters which have been used in the present work are the quantities $\Delta n_0$ and $\Delta n_1$ defined in Section II. It is assumed that each block is isotropic but nonuniform, with a small constant gradient of the refractive index (or the permittivity) in an arbitrary direction. We now consider how the parameters of such a block might be deduced from cavity resonator measurements.

The average index of the block is taken as the mean of the twelve measurements corresponding to the different orientations of the block in the oversize cavity; $\Delta n_0$ is the difference between the average index and the index called for by the lens design. The theoretical dependence of resonant frequency on the permittivity of a uniform block in the cavity can be computed numerically using the Rayleigh-Ritz variational procedure.

The difference $\Delta n_1$ between the average indices of the "heaviest" and "lightest" halves of the block cannot be determined quite so directly, since the direction of dielectric gradient is not necessarily parallel to any edge of the cube. An experimentally measurable quantity, however, is the "half-block spread" H, i.e., the difference between the maximum

## Table I — The Scattering Functions $\varphi_0(x)$ and $\varphi_1(x)$

| $x$ | $\varphi_0(x)$ | $\varphi_1(x)$ | $x$ | $\varphi_0(x)$ | $\varphi_1(x)$ |
|---|---|---|---|---|---|
| 0.0 | 0. | 0. | | | |
| 0.1 | 0.00029512 | 0.00000024 | 6.1 | 2.8440 | 0.85496 |
| 0.2 | 0.0023328 | 0.00000746 | 6.2 | 2.8966 | 0.87218 |
| 0.3 | 0.0077182 | 0.00005556 | 6.3 | 2.9492 | 0.88947 |
| 0.4 | 0.017795 | 0.00022784 | 6.4 | 3.0016 | 0.90692 |
| 0.5 | 0.033547 | 0.00067025 | 6.5 | 3.0538 | 0.92456 |
| 0.6 | 0.055537 | 0.0015960 | 6.6 | 3.1057 | 0.94243 |
| 0.7 | 0.083883 | 0.0032751 | 6.7 | 3.1574 | 0.96048 |
| 0.8 | 0.11828 | 0.0060134 | 6.8 | 3.2088 | 0.97869 |
| 0.9 | 0.15805 | 0.010125 | 6.9 | 3.2601 | 0.99697 |
| 1.0 | 0.20225 | 0.015896 | 7.0 | 3.3113 | 1.0153 |
| 1.1 | 0.24975 | 0.023547 | 7.1 | 3.3624 | 1.0335 |
| 1.2 | 0.29940 | 0.033205 | 7.2 | 3.4136 | 1.0515 |
| 1.3 | 0.35009 | 0.044880 | 7.3 | 3.4649 | 1.0694 |
| 1.4 | 0.40087 | 0.058451 | 7.4 | 3.5164 | 1.0871 |
| 1.5 | 0.45105 | 0.073672 | 7.5 | 3.5680 | 1.1045 |
| 1.6 | 0.50021 | 0.090192 | 7.6 | 3.6197 | 1.1219 |
| 1.7 | 0.54820 | 0.10758 | 7.7 | 3.6715 | 1.1391 |
| 1.8 | 0.59515 | 0.12537 | 7.8 | 3.7234 | 1.1564 |
| 1.9 | 0.64140 | 0.14311 | 7.9 | 3.7752 | 1.1737 |
| 2.0 | 0.68742 | 0.16039 | 8.0 | 3.8269 | 1.1910 |
| 2.1 | 0.73375 | 0.17689 | 8.1 | 3.8784 | 1.2085 |
| 2.2 | 0.78091 | 0.19243 | 8.2 | 3.9298 | 1.2261 |
| 2.3 | 0.82932 | 0.20695 | 8.3 | 3.9811 | 1.2439 |
| 2.4 | 0.87929 | 0.22054 | 8.4 | 4.0322 | 1.2617 |
| 2.5 | 0.93092 | 0.23338 | 8.5 | 4.0831 | 1.2796 |
| 2.6 | 0.98416 | 0.24580 | 8.6 | 4.1341 | 1.2975 |
| 2.7 | 1.0388 | 0.25812 | 8.7 | 4.1850 | 1.3154 |
| 2.8 | 1.0944 | 0.27073 | 8.8 | 4.2359 | 1.3331 |
| 2.9 | 1.1507 | 0.28396 | 8.9 | 4.2870 | 1.3507 |
| 3.0 | 1.2072 | 0.29809 | 9.0 | 4.3381 | 1.3682 |
| 3.1 | 1.2634 | 0.31328 | 9.1 | 4.3893 | 1.3855 |
| 3.2 | 1.3190 | 0.32961 | 9.2 | 4.4406 | 1.4028 |
| 3.3 | 1.3739 | 0.34702 | 9.3 | 4.4920 | 1.4200 |
| 3.4 | 1.4278 | 0.36537 | 9.4 | 4.5434 | 1.4371 |
| 3.5 | 1.4809 | 0.38443 | 9.5 | 4.5947 | 1.4544 |
| 3.6 | 1.5333 | 0.40393 | 9.6 | 4.6460 | 1.4716 |
| 3.7 | 1.5851 | 0.42357 | 9.7 | 4.6972 | 1.4890 |
| 3.8 | 1.6367 | 0.44308 | 9.8 | 4.7482 | 1.5064 |
| 3.9 | 1.6883 | 0.46225 | 9.9 | 4.7992 | 1.5240 |
| 4.0 | 1.7401 | 0.48094 | 10.0 | 4.8500 | 1.5416 |
| 4.1 | 1.7923 | 0.49907 | 10.1 | 4.9008 | 1.5592 |
| 4.2 | 1.8450 | 0.51667 | 10.2 | 4.9515 | 1.5768 |
| 4.3 | 1.8981 | 0.53382 | 10.3 | 5.0023 | 1.5944 |
| 4.4 | 1.9517 | 0.55069 | 10.4 | 5.0530 | 1.6119 |
| 4.5 | 2.0055 | 0.56743 | 10.5 | 5.1039 | 1.6293 |
| 4.6 | 2.0594 | 0.58424 | 10.6 | 5.1548 | 1.6466 |
| 4.7 | 2.1133 | 0.60127 | 10.7 | 5.2058 | 1.6639 |
| 4.8 | 2.1669 | 0.61864 | 10.8 | 5.2568 | 1.6810 |
| 4.9 | 2.2201 | 0.63640 | 10.9 | 5.3079 | 1.6982 |
| 5.0 | 2.2729 | 0.65456 | 11.0 | 5.3590 | 1.7153 |
| 5.1 | 2.3252 | 0.67307 | 11.1 | 5.4100 | 1.7324 |
| 5.2 | 2.3772 | 0.69182 | 11.2 | 5.4610 | 1.7496 |
| 5.3 | 2.4288 | 0.71070 | 11.3 | 5.5119 | 1.7668 |
| 5.4 | 2.4803 | 0.72955 | 11.4 | 5.5627 | 1.7842 |
| 5.5 | 2.5318 | 0.74827 | 11.5 | 5.6135 | 1.8015 |
| 5.6 | 2.5833 | 0.76674 | 11.6 | 5.6641 | 1.8190 |
| 5.7 | 2.6350 | 0.78491 | 11.7 | 5.7148 | 1.8364 |
| 5.8 | 2.6869 | 0.80278 | 11.8 | 5.7654 | 1.8539 |
| 5.9 | 2.7391 | 0.82036 | 11.9 | 5.8160 | 1.8713 |
| 6.0 | 2.7915 | 0.83772 | 12.0 | 5.8666 | 1.8886 |

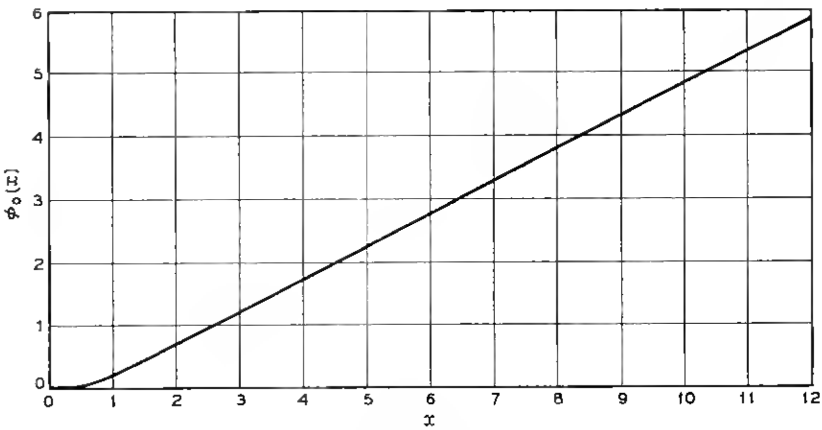


Fig. 1 — The scattering function $\varphi_0(x)$.

and minimum of the twelve measurements of apparent index for each block, taken in all possible orientations. Intuitively one feels that the half-block spread of a linearly-varying block must be approximately proportional to $\Delta n_1$. To get an estimate of the proportionality factor we proceed as follows.

First let the cavity contain a uniform, isotropic block of permittivity $\epsilon_0$. Let the corresponding resonant frequency be $f_0$ and let the electric field be $\mathbf{E}_0(x,y)$, perpendicular to the broad faces of the cavity. If the relative permittivity is nearly unity, then $\mathbf{E}_0$ is nearly the field of the lowest mode in the empty cavity. Coefficients of the expansion of $\mathbf{E}_0$ in terms of the modes of the empty cavity can be obtained numerically, if desired, for any given permittivity by the Rayleigh-Ritz method.

In any case, if the permittivity of the block is now taken to be $\epsilon_0 + \epsilon_1$,

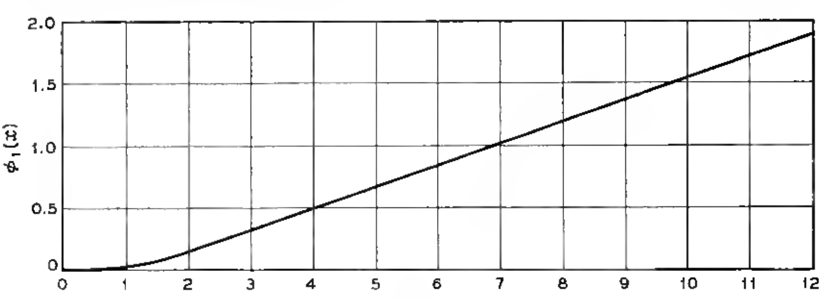


Fig. 2 — The scattering function $\varphi_1(x)$.

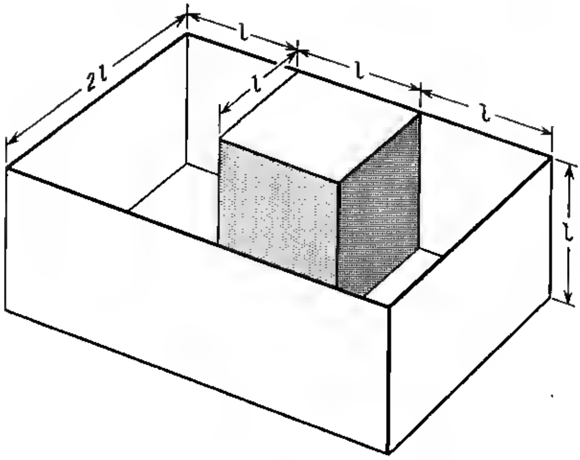


Fig. 3 — Oversize resonant cavity for testing dielectric blocks.

where $\epsilon_1$ is an arbitrary small dielectric deviation, it is well known[8] that the shift in resonant frequency is given approximately by

$$\frac{\Delta f}{f_0} = -\frac{\int \epsilon_1 E_0^2 \, dV}{2\int \epsilon E_0^2 \, dV} . \tag{33}$$

The upper integral in (33) is taken over the volume of the block, since $\epsilon_1$ vanishes elsewhere. The lower integral is taken over the volume of the whole cavity, with $\epsilon = \epsilon_0$ in the block and $\epsilon = \epsilon_v$ (free space) elsewhere. Equation (33) could be generalized to apply to anisotropic media if desired.

Let us define the effective index deviation $\Delta n_e$ to be equal to the uniform deviation which would give the same shift from the frequency associated with the nominal block. From (33),

$$\Delta n_e = \frac{\int \Delta n E_0^2 \, dV}{\int E_0^2 \, dV} , \tag{34}$$

where $\Delta n$ is the pointwise deviation from the nominal value and both integrals are taken over the volume of the block. We shall consider two cases: (a) a linearly varying block with the index gradient parallel to one edge, and (b) a linearly varying block with the index gradient

parallel to a body diagonal. In the present calculation we shall take $\mathbf{E}_0$ to be the field of the lowest mode of the empty cavity; the extra labor involved in using a more accurate field would not be justified, because of the crudeness of the approximations which have already been made in treating the scattering problem. If the origin is taken at the center of the cavity, so that the block fills the space $0 \leqq x \leqq l$, $|y| \leqq \frac{1}{2}l$, $|z| \leqq \frac{1}{2}l$, then the electric field is in the $z$-direction and is proportional to

$$E_0 = \cos\frac{\pi x}{2l}\cos\frac{\pi y}{3l}. \tag{35}$$

If the index gradient is in the $x$-direction and $\Delta n_1$ is the difference between the average indices of the heaviest and lightest halves of the block, then

$$\Delta n = \frac{\Delta n_1}{\frac{1}{2}l}\left(\tfrac{1}{2}l - x\right). \tag{36}$$

The maximum effective index deviation is

$$\Delta n_e = \frac{\dfrac{\Delta n_1}{\frac{1}{2}l}\displaystyle\int_0^l \left(\tfrac{1}{2}l - x\right)\cos^2\frac{\pi x}{2l}\,dx}{\displaystyle\int_0^l \cos^2\frac{\pi x}{2l}\,dx} = \frac{4\Delta n_1}{\pi^2}, \tag{37}$$

corresponding to a half-block spread of

$$H = \frac{8\Delta n_1}{\pi^2} = 0.811\Delta n_1. \tag{38}$$

When the index gradient is parallel to a body diagonal of the block, the calculation is a little more complicated. We shift the origin to the center of the block, so that the block occupies the space $|x'| \leqq \frac{1}{2}l$, $|y'| \leqq \frac{1}{2}l$, $|z'| \leqq \frac{1}{2}l$, and write the electric field as

$$E_0 = \cos\frac{\pi(x' + \frac{1}{2}l)}{2l}\cos\frac{\pi y'}{3l}. \tag{39}$$

Let the index deviation be

$$\Delta n = -cs = -\frac{c(x' + y' + z')}{\sqrt{3}}, \tag{40}$$

where $c$ is a constant and

$$\frac{x' + y' + z'}{\sqrt{3}} = s \tag{41}$$

is the equation of a plane normal to the diagonal $x' = y' = z'$ and at a distance $s$ from the origin. The area of the part of the plane which lies inside the cubical block can be shown by a little geometry to be

$$A(s) = \begin{cases} 3\sqrt{3}(\tfrac{1}{4}l^2 - s^2), & 0 \leqq s \leqq \tfrac{1}{2}l/\sqrt{3}, \\ \dfrac{3\sqrt{3}}{2}(\tfrac{1}{2}\sqrt{3}l - s)^2, & \tfrac{1}{2}l/\sqrt{3} \leqq s \leqq \tfrac{1}{2}\sqrt{3}l. \end{cases} \tag{42}$$

The difference between the average index of the heaviest and lightest halves of the block is

$$\Delta n_1 = \frac{4}{l^3}\int_0^{\frac{1}{2}\sqrt{3}l} csA(s)ds = \frac{13\sqrt{3}cl}{48}. \tag{43}$$

From (34), the effective index deviation is

$$\Delta n_e = -\frac{\displaystyle\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} \frac{c(x'+y'+z')}{\sqrt{3}} \cos^2\frac{\pi(x'+\frac{1}{2}l)}{2l}\cos^2\frac{\pi y'}{3l}\,dx'\,dy'\,dz'}{\displaystyle\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} \cos^2\frac{\pi(x'+\frac{1}{2}l)}{2l}\cos^2\frac{\pi y'}{3l}\,dx'\,dy'\,dz'} \tag{44}$$

$$= \frac{2cl}{\pi^2\sqrt{3}} = \frac{32\Delta n_1}{13\pi^2},$$

where we have used (43) to express $cl$ in terms of $\Delta n_1$. It follows that the half-block spread is

$$H = \frac{64\Delta n_1}{13\pi^2} = 0.499\Delta n_1. \tag{45}$$

It is reasonable to expect that, whatever the orientation of index gradient in a linearly varying block, the proportionality constant relating $\Delta n_1$ to $H$ will lie between the values given by (38) and (45). When in the present approximate treatment it is necessary to express $\Delta n_1$ in terms of $H$, we shall take an intermediate value of the coefficient and set

$$H \approx 0.625\Delta n_1, \qquad \Delta n_1 \approx 1.6H. \tag{46}$$

In view of the universal temptation to substitute numbers into any formula which appears to be written in terms of measured quantities, it must be emphasized that the linearly varying block model used in the foregoing analysis is *known to be incorrect*, at least for sufficiently large blocks produced by current manufacturing techniques. Experiments in which 2-foot cubes of loaded polystyrene foam were sawed up into smaller cubes and individually measured have shown the presence of

marked short-range fluctuations in permittivity which contribute very little to the half-block spread. If the properties of a typical block do not vary linearly but in a more complicated way with position inside the block, it is difficult to know what, if any, significance to attach to the "effective scattering loss tangent" given by (29).

## V. NUMERICAL EXAMPLES

We shall use the present theory to compute some numerical examples of scattering loss in Luneberg lenses, in order to get an idea of the size of the numbers involved, even though we have just observed that the linearly varying block may in many cases be an unrealistic model.

If we assume that the probability distribution of $\Delta n_0$ is uniform between $-(\Delta n_0)_{\max}$ and $+(\Delta n_0)_{\max}$, and that $H$ is uniformly distributed between 0 and $H_{\max}$, then expressing $\Delta n_1$ approximately in terms of $H$ by (46), we find that (29) becomes

$$\tan \delta = \frac{(\Delta n_0)_{\max}^2}{n_0^2} \varphi_0(x) + 1.5 \frac{H_{\max}^2}{n_0^2} \varphi_1(x), \tag{47}$$

where

$$x = 2\pi a n_0/\lambda_v , \tag{48}$$

$$a = 0.620l, \tag{49}$$

and $\lambda_v$ is the vacuum wavelength.

In a companion paper,[9] numerical formulas have been given for the attenuation of electromagnetic energy by a uniformly illuminated Luneberg lens in which the loss tangent of the lens material is any linear function of the refractive index. In the reference cited, the loss tangent was supposed to be due to dissipation, but it can equally well be due to scattering so far as the effective power loss is concerned. It is assumed that the loss tangent can be written in the form

$$\tan \delta = An + B, \tag{50}$$

where $A$ and $B$ are constants determined by passing a line through the values of tan $\delta$ corresponding to the surface and the center of the lens, or by a least-squares fit to more than two points if desired. Then, for example, the fractional power loss in a lens of radius $R$, whose focal point is at a distance $0.1R$ outside the surface, is given by

$$\frac{\Delta P}{P_0} = \frac{R}{\lambda_v} [15.46A + 13.13B]. \tag{51}$$

In the following examples we shall assume that $(\Delta n_0)_{max}$ and $H_{max}$ are constant throughout the lens. For any given ratio of block size to wavelength, it is simple to calculate $\tan \delta$ at the surface of the lens ($n = 1$) and at the center ($n = 1.36$) from (47), and then to determine the values of $A$ and $B$ in (50). Finally the fractional power loss may be written in the form

$$\frac{\Delta P}{P_0} = \frac{R}{\lambda_v} [F_0(\Delta n_0)_{max}{}^2 + F_1 H_{max}{}^2]. \tag{52}$$

The dimensionless coefficients $F_0$ and $F_1$ are given as functions of $l/\lambda_v$ in Table II, and are plotted in Fig. 4.

As a first example, consider a lens 40 wavelengths in diameter, made of one-wavelength dielectric cubes. If we take

$$\begin{aligned} R/\lambda_v &= 20, \\ l/\lambda_v &= 1, \\ (\Delta n_0)_{max} &= 0.005, \\ H_{max} &= 0.020, \end{aligned} \tag{53}$$

then (52) and Table II yield

$$\Delta P/P_0 = 0.077 \quad \text{or} \quad 0.35 \text{ db loss}. \tag{54}$$

The loss would be the same if the frequency were doubled and the block size halved, while the lens diameter and the values of $(\Delta n_0)_{max}$ and $H_{max}$ were held constant.

TABLE II—THE FUNCTIONS $F_0$ AND $F_1$

| $l/\lambda_v$ | $F_0$ | $F_1$ | $l/\lambda_v$ | $F_0$ | $F_1$ |
|---|---|---|---|---|---|
| 0.0 | 0. | 0. | | | |
| 0.1 | 0.248 | 0.007 | 1.1 | 22.05 | 9.553 |
| 0.2 | 1.544 | 0.160 | 1.2 | 24.42 | 10.69 |
| 0.3 | 3.625 | 0.765 | 1.3 | 26.76 | 11.89 |
| 0.4 | 5.798 | 1.777 | 1.4 | 29.05 | 13.12 |
| 0.5 | 7.960 | 2.837 | 1.5 | 31.36 | 14.31 |
| 0.6 | 10.23 | 3.833 | 1.6 | 33.67 | 15.47 |
| 0.7 | 12.59 | 4.848 | 1.7 | 35.97 | 16.65 |
| 0.8 | 15.01 | 5.906 | 1.8 | 38.25 | 17.84 |
| 0.9 | 17.42 | 7.075 | 1.9 | 40.53 | 19.02 |
| 1.0 | 19.73 | 8.344 | 2.0 | 42.82 | 20.19 |

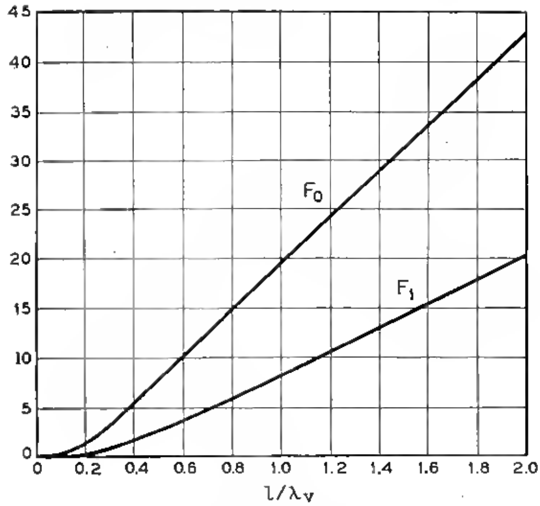


Fig. 4 — The functions $F_0$ and $F_1$.

As a second example, consider a lens with

$$\begin{aligned} R/\lambda_v &= 40, \\ l/\lambda_v &= 2, \\ (\Delta n_0)_{\max} &= 0.005, \\ H_{\max} &= 0.020. \end{aligned} \tag{55}$$

The loss computed from (52) and Table II is

$$\Delta P/P_0 = 0.37 \quad \text{or} \quad 2.0 \text{ db loss.} \tag{56}$$

This would correspond to the lens treated in the first example if it were used at double the frequency with no alterations in physical structure. Of course it is hardly legitimate to apply (52), which was derived on the assumption that the scattering loss is a small perturbation, to predict a loss of 2 db. We *can* conclude from (52), however, that to reduce the scattering loss in the second lens to 0.35 db, it would be necessary to hold $H_{\max}$ down to 0.0065 if $(\Delta n_0)_{\max}$ were still equal to 0.005.

Finally we note that the examples just given refer to blocks whose dimensions are comparable to or greater than a wavelength, and for which the assumption of linear index variation is very likely to be invalid. If the formulas were applied to blocks of fractional wavelength

size, as can easily be done using Table I or Table II, the half-block spread might be a more significant parameter, since very short-range fluctuations scatter so little energy, and the computed results might be more meaningful. The working out of additional numerical examples, however, is left to the reader.

## VI. ACKNOWLEDGMENTS

I am indebted to W. A. Yager for numerous discussions of the problems of losses in foam Luneberg lenses. Miss M. C. Gray derived the expressions (30) and (31) for $\varphi_0(x)$ and $\varphi_1(x)$, and Miss P. A. Hamilton computed the tables.

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